Optimal Decision Tree Size vs. Minimal Disjoint CDNF size for a specific Boolean Function of Degree 6

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Bshouty shows that for any two boolean functions $f : \{0,1\}^{n1} \rightarrow 0,1$ and $g : \{0,1\}^{n2} \rightarrow \{0,1\}$ and two sets of disjoint variables $x = (x_1, ...x_{n1})$ and $y = (y_1, ...y_{n2})$ we have

$$size_{DT}(f(x) \oplus g(y)) = size_{DT}(f(x)) \cdot size_{DT}(g(y))$$

Bshouty also shows that for any two boolean functions $f : \{0,1\}^{n1} \rightarrow \{0,1\}$ and $g : \{0,1\}^{n2} \rightarrow \{0,1\}$ and two sets of disjoint variables $x = (x_1, ...x_{n1})$ and $y = (y_1, ...y_{n2})$ we have

$$size_{DCD}(f(x) \oplus g(y)) \leq size_{DCD}(f(x)) \cdot size_{DCD}(g(y))$$

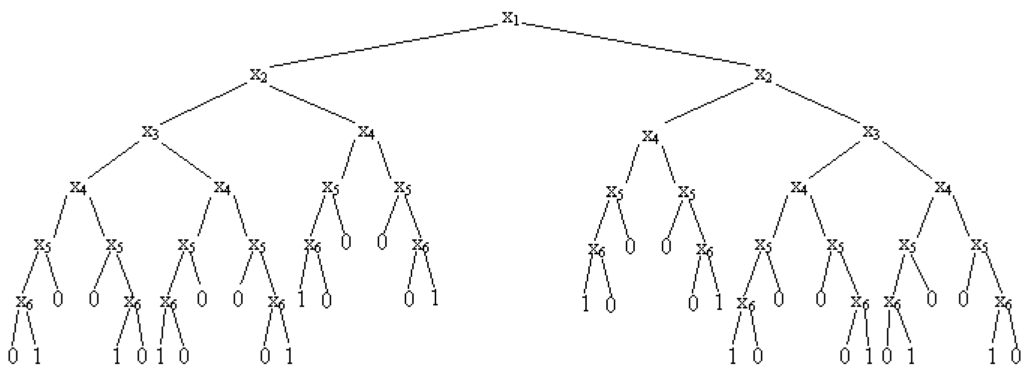
I have previously shown that the boolean function on three variables $f(x_1, x_2, x_3) = x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}$ has a minimal disjoint CDNF of size 5 and an optimal decison tree of size 6. I will now determine the size of a minimal disjoint CDNF and an optimal decision tree for $f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$, which is the function that is the exclusive or of the previous function with itself on two sets of disjoint variables. In doing so, I am looking to support or disprove my conjecture that if any Boolean function on n variables can be represented by a minimal disjoint CDNF of size s, then the size of its corresponding optimal decision tree is at most $s^{\frac{\log 6}{\log 5}}$.

truth table for $(x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$

x_1	x_2	x_3	x_4	x_5	x_6	$x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}$	$x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6}$	$(x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$
0	0	0	0	0	0	1	1	
0	0	0	0	0	1	1		1
0	0	0	0	1	0	1		1
0	0	0	0	1	1	1		1
0	0	0	1	0	0	1		1
0	0	0	1	0	1	1		1
0	0	0	1	1	0	1		1
0	0	0	1	1	1	1	1	
0	0	1	0	0	0		1	1
0	0	1	0	0	1			
0	0	1	0	1	0			
0	0	1	0	1	1			
0	0	1	1	0	0			
0	0	1	1	0	1			
0	0	1	1	1	0			
0	0	1	1	1	1		1	1
0	1	0	0	0	0		1	1
0	1	0	0	0	1			
0	1	0	0	1	0			
0	1	0	0	1	1			
0	1	0	1	0	0			
0	1	0	1	0	1			
0	1	0	1	1	0			
0	1	0	1	1	1		1	1
0	1	1	0	0	0		1	1
0	1	1	0	0	1			
0	1	1	0	1	0			
0	1	1	0	1	1			
0	1	1	1	0	0			
0	1	1	1	0	1			
0	1	1	1	1	0			
0	1	1	1	1	1		1	1
1	0	0	0	0	0		1	1
1	0	0	0	0	1			
1	0	0	0	1	0			
1	0	0	0	1	1			
1	0	0	1	0	0			
1	0	0	1	0	1			
1	0	0	1	1	0			
1	0	0	1	1	1		1	1
1	0	1	0	0	0		1	1
1	0	1	0	0	1			
1	0	1	0	1	0			
1	0	1	0	1	1			
1	0	1	1	0	0			
1	0	1	1	0	1			
1	0	1	1	1	0			
1	0	1	1	1	1		1	1
1	1	0	0	0	0		1	1
1	1	0	0	0	1			
1	1	0	0	1	0			
1	1	0	0	1	1			
1	1	0	1	0	0			
1	1	0	1	0	1			
1	1	0	1	1	0			
1	1	0	1	1	1		1	1
1	1	1	0	0	0	1	1	
1	1	1	0	0	1	1		1
1	1	1	0	1	0	1		1
1	1	1	0	1	1	1		1
1	1	1	1	0	0	1		1
1	1	1	1	0	1	1		1
1	1	1	1	1	0	1		1
1	1	1	1	1	1	1	1	

x_1 x_2 x_3 \ x_4 x_5 x_6	1 1 1	1 1 0	1 0 0	1 0 1	0 0 1	0 0 0	0 1 0	0 1 1
1 1 1		1	1	1	1		1	1
1 1 0	1					1		
1 0 0	1					1		
1 0 1	1					1		
0 0 1	1					1		
0 0 0		1	1	1	1		1	1
0 1 0	1					1		
0 1 1	1					1		

Figure 1: Karnaugh map for $(x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$



$$(x_1x_2x_3x_4\bar{x_6} + x_1x_2x_3\bar{x_5}x_6 + x_1x_2x_3\bar{x_4}x_5 + x_1\bar{x_3}x_4x_5x_6 + \bar{x_2}x_3x_4x_5x_6 + \bar{x_1}x_2x_4x_5x_6 + \bar{x_1}\bar{x_2}\bar{x_3}x_4\bar{x_6} + \bar{x_1}\bar{x_2}\bar{x_3}\bar{x_5}x_6 + \bar{x_1}\bar{x_2}\bar{x_3}\bar{x_4}x_5 + x_1\bar{x_3}\bar{x_4}\bar{x_5}\bar{x_6} + \bar{x_2}x_3\bar{x_4}\bar{x_5}\bar{x_6} + \bar{x_1}x_2\bar{x_4}\bar{x_5}\bar{x_6},\ x_1x_2x_3x_4x_5x_6 + x_1x_2x_3\bar{x_4}\bar{x_5}\bar{x_6} + x_1\bar{x_3}x_4\bar{x_6} + x_1\bar{x_3}\bar{x_5}x_6 + x_1\bar{x_3}\bar{x_4}x_5 + \bar{x_2}x_3x_4\bar{x_6} + \bar{x_2}x_3\bar{x_5}x_6 + \bar{x_2}x_3\bar{x_4}x_5 + \bar{x_1}\bar{x_2}\bar{x_3}x_4x_5x_6 + \bar{x_1}\bar{x_2}\bar{x_3}\bar{x_4}\bar{x_5}\bar{x_6} + \bar{x_1}x_2x_4\bar{x_6} + \bar{x_1}x_2\bar{x_5}x_6 + \bar{x_1}x_2\bar{x_4}x_5)$$

Figure 2: disjoint CDNF and decision tree for $(x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$

Therefore, $f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1x_2x_3 + \bar{x_1}\bar{x_2}\bar{x_3}) \oplus (x_4x_5x_6 + \bar{x_4}\bar{x_5}\bar{x_6})$ has a minimal disjoint CDNF of size 25 and an optimal decision tree of size 36. This conclusion supports Bshouty's findings as well as my conjecture that if any Boolean function on n variables can be represented by a minimal disjoint CDNF of size s, then the size of its corresponding optimal decision tree is at most $s^{\frac{\log 6}{\log 5}}$.